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MODE III CRACK IN A LAMINATED MEDIUM

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Abstract— Antiplane deformation of a laminated medium with a semi-infinite crack parallel to the interfaces between the layers is considered. The homogeneous, elastic and isotropic layers of two different types are arranged periodically. This allows the reduction of the initial boundary value problem to the Wiener–Hopf equation by combined application of the Laplace and the discrete Fourier transforms. For the specific case of an exponentially decaying load the solution is obtained in closed form by means of rapidly convergent triple integrals. The simple expressions for the stress intensity factor and the energy release rate are derived. For some limiting cases the solution is found to be consistent with previous results. A parametric study gave an opportunity to examine the accuracy of the known approximate sandwich model of multilayered composites. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Multilayered laminates with a large number of layers represent an important type of composites employed recently in engineering practice. Analysis of the overall mechanical properties of these composites has been developed to an advanced level. Deriving the laminate stiffnesses can be easily carried out for any number of layers. But when fracture of a laminate is investigated and exact strongly non-uniform stress distribution is required, increasing the number of plies essentially complicates the analysis. An approximate discrete model of the layered media was employed by Slepyan (1974) who considered the steady-state crack propagation. Another way to avoid difficulties arising in the study of fracture of the multilayered laminates is to use models with a reduced number of layers. Different models of this type are employed in fracture mechanics of composites. A finite mode 3 interface crack between inner layers of a four-layered composite was investigated by Chen and Sih (1971). A large number of layers in the actual laminate was averaged by the two outer layers of infinite height. Ashbaugh (1973) considered a mode 1 tunneling crack in a bimaterial laminated composite. In that work the multilayered composite is replaced by a three-layered system where the cracked layer of one material is sandwiched between two half-spaces of the second material. Thus it is supposed that all the perturbation generated by the crack is localized in the cracked layer and the two neighboring ones. The same three layered model was employed by Hilton and Sih (1971) who considered different crack locations in the middle layer under the assumption that the outer layers possess some effective elastic properties. Numerous results for cracks in different sandwich systems may be found in Hutchinson and Suo (1991) and Sih and Chen (1981).

An investigation of the stress field near the crack tip in a multilayered composite consisting of thin plies leads to the necessity of introducing models with a large number of layers. The term "thin" means here that the ratio of the crack length to the characteristic ply thickness is not too small. Employing for such laminates the models with a reduced number of layers is possible only in the special case of "short" loadings when the characteristic ply thickness sufficiently exceeds the length parameter associated with the load distribution. This statement will be illustrated by some results in the present paper.

A bimaterial medium consisting of an infinite number of layers with a semi-infinite mode III crack in one of them, to be considered below, is an example of the mentioned multilayered model. The materials of the layers are taken as homogeneous elastic and isotropic. In Section 2 a boundary problem for unbounded medium is formulated. Then by

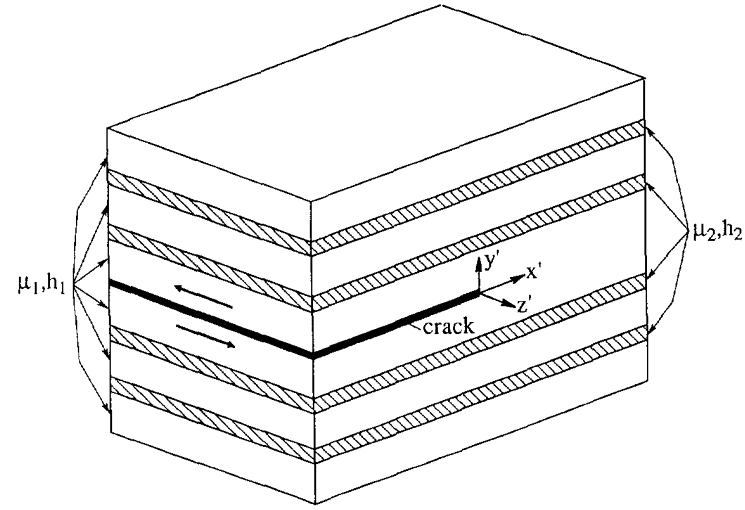

Fig. 1. Antiplane deformation of a periodically laminated medium with a semi-infinite crack at the plane of symmetry.

using the regular arrangement of the layers the initial problem is reduced to a problem for a representative cell with an additional mixed boundary condition. This is accomplished with the help of the discrete Fourier transform. The closed form solution of the problem is constructed in Section 3. Application of the Laplace transform leads to a Wiener-Hopf equation which is solved exactly by means of Cauchy type integrals. In the case of an exponentially decaying load the final expressions for the components of the stress-strain field in any point of multilayered composite are obtained in the form of rapidly convergent triple integrals. In Section 4 formulae for the stress intensity factor and the energy release rate are derived and a parametric study with graphical illustration is carried out. The obtained exact results are compared with those given by "sandwich" approximation. Some limiting cases of interest are considered.

2. FORMULATION OF THE PROBLEM FOR A REPRESENTATIVE CELL

Consider a bimaterial composite consisting of an infinite number of isotropic elastic layers. Layers of two different types characterized by thicknesses h_r and elastic shear moduli μ_r $(r = 1, 2)$ are arranged symmetrically with respect to the plane $y' = 0$ of the global coordinate system x', y', z'. Hence one layer of double thickness $-h_1 < y' < h_1$ is produced (Fig. 1). The material of this layer having shear modulus μ_1 will be regarded as the inner material and the second material with modulus μ_2 as the outer one. A semi-infinite crack is located in the plane of symmetry of the body $y' = 0$. Shear loading $q(x')$, $x' < 0$ applied to its faces produces the antiplane stress state.

Since the problem is skew-symmetric it is sufficient to consider a boundary value problem for the half-space $y' > 0$ (Fig. 2). Note that such a problem may also be employed

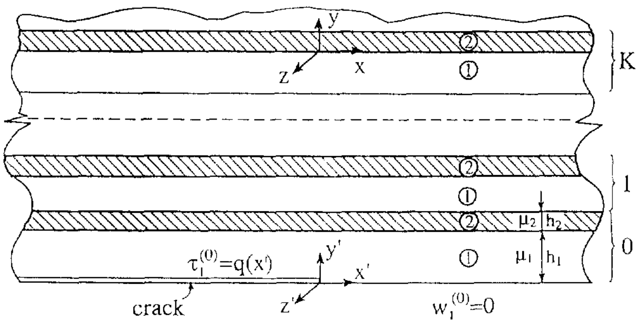

Fig. 2. Equivalent problem for a layered half-space debonding from a rigid substrate.

in the study of the debonding of a multilayered laminate from a rigid substrate. The half-space will be viewed as an assemblage of an infinite number of identical cells $(h_1+h_2)k < y' < (h_1+h_2)(k+1)$, $k = 0, 1, 2, \ldots$ Each cell having thickness h_1+h_2 includes two layers of both types. In the local coordinate system x, y, z

$$x \equiv x', \quad y = y'-k(h_1+h_2)-h_1, \quad z \equiv z' \tag{1}$$

these layers occupy the regions $-h_1 < y < 0$ (type one) and $0 < y < h_2$ (type two). Since the deformation is antiplane only displacements in the z-direction take place

$$w_r^{(k)} = w_r^{(k)}(x,y) \quad r = 1,2; \quad k = 0,1,2,\ldots \tag{2}$$

and the boundary value problem for the layered elastic half-space can be formulated as follows. The only non-zero stresses are given by the relations

$$\tau_{r_{xz}}^{(k)} = \mu_r \frac{\partial w_r^{(k)}}{\partial x}, \quad \tau_{r_{yz}}^{(k)} = \mu_r \frac{\partial w_r^{(k)}}{\partial y}. \tag{3}$$

For the sake of brevity the subscripts yz hereafter will be omitted: $\tau_{r_{yz}}^{(k)} \equiv \tau_r^{(k)}$. To fulfill the equations of equilibrium it is sufficient to satisfy the harmonic equation

$$\frac{\partial^2 w_r^{(k)}}{\partial x^2} + \frac{\partial^2 w_r^{(k)}}{\partial y^2} = 0. \tag{4}$$

The continuity conditions at the interfaces are expressed as the conditions between the layers of each cell

$$\tau_2^{(k)}(x,0) - \tau_1^{(k)}(x,0) = 0 \tag{5}$$

$$w_2^{(k)}(x,0) - w_1^{(k)}(x,0) = 0 \tag{6}$$

and the conditions at the boundaries between the cells

$$\tau_1^{(k+1)}(x,-h_1) - \tau_2^{(k)}(x,h_2) = 0 \tag{7}$$

$$w_1^{(k+1)}(x,-h_1) - w_2^{(k)}(x,h_2) = 0, \tag{8}$$

where $-\infty < x < \infty$, and $k = 0, 1, 2, \ldots$. The mixed condition at the lower boundary of the cell with $k = 0$ is defined by the shear loading applied to the crack face

$$\tau_1^{(0)}(x,-h_1) = q(x), \quad -\infty < x < 0, \tag{9}$$

and by the absence of displacements in front of the crack

$$w_1^{(0)}(x,-h_1) = 0, \quad 0 < x < \infty. \tag{10}$$

The structure of the formulated boundary value problem (2)–(10) calls rather naturally for the use of the discrete Fourier transform. But its direct application is impossible since the distribution of the elastic properties in the domain does not possess the translational symmetry in the y'-direction.

A means to overcome this obstacle was indicated by Nuller (1981). Assume that the stresses $\tau_1^{(0)}(-h_1,x)$ existing in front of the crack in the domain $y' = 0$, $x > 0$ are defined by some function $Q(x)$. Let us now replace the rigid substrate in the domain $y' < 0$ by a layered elastic half space assuming that the relations (1)–(8) are valid also for $k = -1$,

$-2, \ldots$. The mating conditions in the plane $y' = 0$ will be generated so that the stress and strain states of the upper half space $y' > 0$ remain the same as before replacement. Note that in contrast to the initial problem, the plane $y' = 0$ becomes an interface between material 1 (lower layer of the cell number 0) and material 2 (upper layer of the cell number -1).

Further, consider an auxiliary problem for the lower half space subjected, at the boundary $y' = 0$, to the loading $Q(x)$ for $x > 0$ and $q(x)$ for $x < 0$. Consequently the displacements of the deformed boundary will be $w_2^{(-1)}(x, h_2) = W(x)$. Here $W(x)$ is some unknown function.

One now observes that the mentioned mating conditions at the interface between upper and lower half spaces are given by the continuity condition for the stresses

$$\tau_1^{(0)}(x, -h_1) - \tau_2^{(-1)}(x, h_2) = 0, \quad -\infty < x < \infty \tag{11}$$

and a condition of the displacement jump

$$w_1^{(0)}(x, -h_1) - w_2^{(-1)}(x, h_2) = W(x), \quad -\infty < x < \infty. \tag{12}$$

Note that the conditions (9), (10) also remain valid. In view of the last relation the boundary conditions for the displacements at the interfaces between all the cells may be rewritten as

$$w_1^{(k+1)}(x, -h_1) - w_2^{(k)}(x, h_2) = \delta_{-1,k} W(x), \quad k = 0, \pm 1, \pm 2, \ldots, \tag{13}$$

where δ_{ij} is the Kronecker delta. Hence the relations (3)–(8) for $k = 0, \pm 1, \pm 2, \ldots$ and condition (13) determine the boundary value problem for the layered space possessing the desired symmetry properties. The stress strain state of the upper half space of this problem is then equivalent to the former one defined by (2)–(10) with $k = 0, 1, 2, \ldots$. Application of the discrete Fourier transform

$$f^*(\phi) = \sum_{k=-\infty}^{k=\infty} f^{(k)} e^{ik\phi}. \tag{14}$$

converts the problem for the whole space to the problem for a representative cell $(-h_1 \leqslant y \leqslant h_2)$:

$$\tau_{rx}^* = \mu_r \frac{\partial w_r^*}{\partial x}, \quad \tau_r^* = \mu_r \frac{\partial w_r^*}{\partial y} \tag{15}$$

$$\frac{\partial^2 w_r^*}{\partial x^2} + \frac{\partial^2 w_r^*}{\partial y^2} = 0 \tag{16}$$

$$\tau_2^*(x, 0, \phi) - \tau_1^*(x, 0, \phi) = 0 \tag{17}$$

$$w_2^*(x, 0, \phi) - w_1^*(x, 0, \phi) = 0 \tag{18}$$

$$\tau_1^*(x, -h_1, \phi) - \gamma \tau_2^*(x, h_2, \phi) = 0 \tag{19}$$

$$w_1^*(x, -h_1, \phi) - \gamma w_2^*(x, h_2, \phi) = W(x), \tag{20}$$

where $\gamma = e^{i\phi}$.

The formal solution of this problem to be constructed in the following section will give all the components of the stress strain state in terms of function $W(x)$. Afterwards the additional mixed boundary condition (9)–(10) will be employed to eliminate this function and to obtain the final solution.

3. METHOD OF SOLUTION

The method for deriving the closed form solution employed here was suggested in the mentioned work by Nuller (1981). Expressions for the functions w_r^* satisfying the equilibrium eqns (16) identically are taken as inverse double-sided Laplace transforms

$$w_r^*(x,y,\phi) = \frac{1}{2\pi}\int_L [A_r(p,\phi)\sin py + B_r(p,\phi)\cos py]\mathrm{e}^{px}\,\mathrm{d}p, \quad r = 1,2 \tag{21}$$

where L is a contour in the plane of complex variable p located in the left hand vicinity of the imaginary axis $\{L\colon \mathrm{Re}\,(p) = -\varepsilon,\ \varepsilon > 0\}$ and A_r, B_r are some functions of the Laplace and Fourier transforms parameters. Substitution of (21) in (15), (17)–(20) yields the expressions for these functions through the Laplace transform of the displacement jump $\bar{W}$

$$A_1(p,\phi) = -\mu(\sin ph_1 + \mu\gamma\sin ph_2)D^{-1}(p,\phi)\bar{W}(p),$$
$$B_1(p,\phi) = \mu(\cos ph_1 - \gamma\cos ph_2)D^{-1}(p,\phi)\bar{W}(p),$$
$$B_2(p,\phi) = B_1(p,\phi), \quad A_2(p,\phi) = \mu^{-1}A_1(p,\phi) \tag{22}$$

where

$$\mu = \frac{\mu_2}{\mu_1} \tag{23}$$

is the elastic moduli ratio, and

$$D(p,\phi) = \mu(\gamma^2+1)+(1+\mu^2)\gamma\sin ph_1\sin ph_2 - 2\mu\gamma\cos ph_1\cos ph_2. \tag{24}$$

Consequently the inverse discrete Fourier transform

$$f^{(k)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(\phi)\mathrm{e}^{-ik\phi}\,\mathrm{d}\phi \tag{25}$$

will give the expressions for all components of the stress-strain field in every cell of the space in terms of $\bar{W}$.

To eliminate function $\bar{W}$ the mixed boundary condition (9)–(10) is employed. Calculation of the displacements and stresses in the crack plane ($k = 0, y = -h_1$) in accordance with (15), (21), (22), (25), substitution into (9)–(10) and exchanging the order of the Laplace and the Fourier transforms leads to the following relations

$$w_1^-(p) = F_1(p)\bar{W}(p) \tag{26}$$

$$\tau^+(p)+\tau^-(p) = F_2(p)\bar{W}(p) \tag{27}$$

Here symbols $w^-(p)$, $\tau^+(p)$ and $\tau^-(p)$ denote the corresponding half Laplace transforms of the displacements at the crack face

$$w^-(p) = \int_{-\infty}^{0} w_1^{(0)}(x,-h_1)e^{-px}\,\mathrm{d}x, \tag{28}$$

of the stresses in front of the crack

$$\tau^+(p) = \int_0^\infty \tau_1^{(0)}(x, -h_1)e^{-px}\,dx \tag{29}$$

and of the applied loading

$$\tau^-(p) = \int_{-\infty}^0 q(x)e^{-px}\,dx. \tag{30}$$

Superscripts "+" and "−" as usual indicate functions analytic in the left, $\mathrm{Re}(p) \leqslant -\varepsilon$, and right, $\mathrm{Re}(p) \geqslant -\varepsilon$, half-planes respectively. The expressions for the functions $F_1(p)$ and $F_2(p)$ are found to be

$$F_1(p) = \frac{\mu}{2\pi}\int_{-\pi}^{\pi} [1+\gamma(\mu \sin ph_1 \sin ph_2 - \cos ph_1 \cos ph_2)]D^{-1}(p,\phi)\,d\phi,$$

$$F_2(p) = -\frac{\mu_2}{2\pi}\int_{-\pi}^{\pi} p\gamma[\mu \cos ph_1 \sin ph_2 + \sin ph_1 \cos ph_2]D^{-1}(p,\phi)\,d\phi. \tag{31}$$

Relations (26) and (27) are equivalent to the Wiener–Hopf equation

$$\tau^+(p) = N(p)W^-(p) - \tau^-(p), \quad p \in L \tag{32}$$

where

$$N(p) = \frac{F_2(p)}{F_1(p)}. \tag{33}$$

The technique employed for its solution is similar to that used by Ryvkin and Sills (1993) and will be just summarized here. The applied traction $q(x)$ is taken in the form

$$q(x) = -q_0 \exp(x/l) \tag{34}$$

where q_0 is a constant with dimensions of stress and l is a length parameter describing the loading decay rate. Then the half-transform (30) is given by

$$\tau^-(p) = \frac{q_0}{p-b} \tag{35}$$

where $b = 1/l$.

Since both functions $N(p)$ and $\tau^-(p)$ have no singularities in the strip $-\varepsilon \leqslant \mathrm{Re}(p) \leqslant 0$ it is possible to move contour L to the imaginary axis $\mathrm{Re}(p) = 0$. Owing to this, the integrals in (31) are evaluated analytically and consequently the problem coefficient N for $p = it$, $-\infty < t < \infty$, is expressed as

$$N(it) = -\mu_1 t B(t), \tag{36}$$

$$B(t) = \frac{[2\mu + (1+\mu^2)t_1 t_2]^2 - 4\mu^2 \cosh^{-2} h_1 t \cosh^{-2} h_2 t^{1/2} + (1-\mu^2)t_1 t_2}{2(\mu t_1 + t_2)},$$

$$t_1 = \tanh h_1 t, \quad t_2 = \tanh h_2 t. \tag{37}$$

So function $N(p)$ on the imaginary axis is real valued and may be easily factorized in terms of Cauchy-type integrals (see Gakhov, 1966)

$$N(p) = \frac{N^-(p)}{N^+(p)}, \tag{38}$$

$$N^\pm(p) = N_1^\pm(p) N_2^\pm(p), \tag{39}$$

$$N_1^+(p) = p^{-1}\sqrt{p+0}, \quad N_1^-(p) = i\mu_1\sqrt{p-0}, \tag{40}$$

$$N_2^\pm(p) = \exp\left[\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln N_2(u)}{u-p}\mathrm{d}u\right], \quad N_2(it) = \operatorname{sgn}(t)B(t), \tag{41}$$

where the expressions $p+0$ and $p-0$ in (40) are to be interpreted as that branch cut for the function $\sqrt{p}$ taken along the negative and positive real axes, respectively. Note that function $N_2(it)$ is continuous at point $t = 0$ since

$$\lim_{t\to\pm 0} B(t) = \pm\sqrt{\frac{\mu(h_1+\mu h_2)}{\mu h_1 + h_2}}.$$

Using the factorization to rearrange (32), applying the generalized Liouville theorem and making the usual physical assumption that the local strain energy in the neighborhood of the crack tip is bounded one obtains the solution of the Wiener–Hopf equation in the following form

$$\tau^+(p) = \frac{q_0}{p-b}\left[\frac{N^+(b)}{N^+(p)} - 1\right], \tag{42}$$

$$W^-(p) = \frac{q_0 N^+(b)}{(p-b)N^-(p)}. \tag{43}$$

It is now possible to determine the unknown transform $\bar{W}(p)$ from (26) or (27). Its substitution into (22) and carrying out the inverse transformations (21), (25) gives the desired closed form solution of the initial problem (2)–(10) as triple integrals. Two of the integrals have infinite limits but their numerical evaluation is easy since, as it is possible to show, the integrands decay exponentially for large values of their respective arguments

$$\lim_{u\to\pm i\infty}\frac{\ln N_2(u)}{p-u} = O[\max(\mathrm{e}^{-h_1|u|}, \mathrm{e}^{-h_2|u|})],$$

$$\lim_{p\to\pm i\infty}\bar{w}_r^*(p,y,\phi) = O[\mathrm{e}^{[(-1)^r y - h_r]|p|}],$$

where function $\bar{w}_r^*(p,y,\phi)$ denotes the Laplace transform of function $w_r^*(x,y,\phi)$.

4. RESULTS

4.1. *Stress intensity factor*

To carry out fracture analysis the behavior of the obtained solution in cell number 0 near the point $x = 0$, $y = 0$ must be examined. Asymptotes of the singular stresses and the crack face displacements in the crack tip vicinity may be found from the asymptotic behavior of the corresponding Laplace transforms for large values of $|p|$. From (42), (43) it follows that for $|p| \to \infty$ in the respective half-planes

$$\tau^-(p) \sim q_0 N^-(b) p^{-1/2}, \tag{44}$$

$$w^-(p) \sim \frac{q_0}{\mu_1} N^+(b)(-p)^{-3/2}. \tag{45}$$

Consequently (Noble, 1958),

$$\tau_1^{(0)}(x, -h_1)_{x \to -0} = \frac{q_0 N^-(b)}{\sqrt{\pi x}} \tag{46}$$

and

$$w_1^{(0)}(x, -h_1)_{x \to -0} = \frac{2q_0 N^+(b)}{\mu_1} \sqrt{\frac{-x}{\pi}}. \tag{47}$$

Deriving the value of $N^+(b)$ in accordance with (39)–(41) and using even properties of the function N_2 one obtains the expression for the stress intensity factor

$$K = q_0 \sqrt{\frac{2}{b}} N_2^+(b) \tag{48}$$

$$N_2^+(b) = \exp\left\{ -\frac{1}{\pi} \int_0^\infty \frac{\ln B(b\xi)}{\xi^2 + 1} \mathrm{d}\xi \right\}. \tag{49}$$

Consider now a simple reference problem of a semi-infinite crack in a homogeneous space possessing the elastic properties associated with the first (inner) material. The loading is the same as (34). To calculate the stress intensity factor for this problem from the obtained solution for a layered media it is sufficient to consider the case of identical materials $\mu = 1$ or, alternatively, to eliminate the volume occupied by the second (outer) material, i.e., to assume that $h_2 \to 0$. In both cases, from (37) follows that $B(t) \equiv 1$. Hence in accordance with (48), (49) the stress intensity factor K_h in this problem is, not surprisingly, found to be (see Atkinson, 1977)

$$K_h = q_0 \sqrt{\frac{2}{b}} \tag{50}$$

Recall that $b = 1/l$. The value of K_h is suitable for normalization of the general result. Then the expression of the non-dimensional stress intensity factor $\hat{K}$ is given by a simple formula

$$\hat{K} = \frac{K}{K_h} = N_2^+(b) \tag{51}$$

where $N_2^+(b)$ must be calculated from (49), (37).

From the dimensional considerations it follows that the obtained non-dimensional stress intensity factor may be viewed as a function of the three non-dimensional parameters of the problem

$$\hat{K} = \hat{K}\left(\mu, \frac{h_2}{h_1}, \frac{h_1}{l}\right). \tag{52}$$

This fact may be also observed explicitly from (37), (49) and (51). It should be noted here that as was emphasized by Ryvkin *et al.* (1994) the actual size of the K-controlled zone in the crack tip vicinity strongly depends upon the properties of the inner layer and degenerates

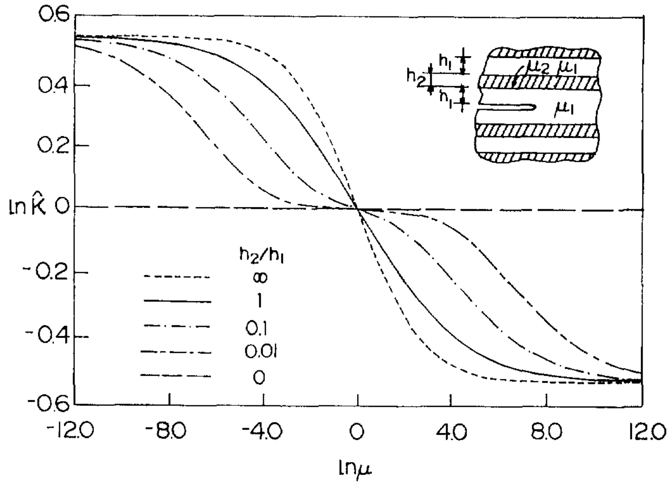

Fig. 3. Graph of the normalized stress intensity factor $\hat{K} = (q_0\sqrt{2l})^{-1}K$ vs the ratio of the layers shear moduli $\mu = \mu_2/\mu_1$ for the crack in the laminated medium. Logarithmic scale is used; $h_1/l = 0.5$. Horizontal line corresponds to the case of the homogeneous medium ($h_2 = 0$).

to zero when $h_1 \to 0$ or $\mu_1 \to 0$. Therefore in further analysis it is reasonable to interpret the variation of non-dimensional parameters as changing in μ_2, h_2 or l when h_1 and μ_1 remain fixed (recall that $\mu = \mu_2/\mu_1$).

In Fig. 3 the dependence of the non-dimensional stress intensity factor from shear moduli ratio μ is illustrated. A family of curves for $h_2/h_1 = 0, 0.01, 0.1, 1, \infty$ with $h_1/l = 0.5$ is presented in a logarithmic scale. The degenerate case $h_2 = 0$ corresponds to the reference problem for the homogeneous space. So in accordance with (51) $\hat{K} \equiv 1$ and the respecting graph is a horizontal straight line. Decreasing of the stress intensity factor with increasing of μ as observed in the rest of the cases is rather plausible. In fact, when μ (i.e. μ_2) increases then the outer layers become stiffer and reinforce the body. Hence the displacements on the outer boundaries of the middle layer containing the crack will be more restricted and consequently the stress intensity factor diminishes. The influence of the thicknesses ratio h_2/h_1 on the behavior of $\hat{K}$ which is observed in the figure may be explained by similar considerations. With respect to the reference problem the effect of reinforcing (weakening) for the fixed $\mu > 1 (\mu < 1)$ amplifies with the growth of h_2, therefore $\hat{K}$ decreases (increases). The extreme curve for $h_2/h_1 = \infty$ presents the important case of the cracked layer sandwiched between two identical half-spaces. For $\mu = 1$ the laminated medium becomes homogeneous irrespective of h_2, consequently $\hat{K} = 1$ for all the curves.

The curves have also the same asymptotes for $\mu \to 0$ ($\ln\mu \to -\infty$) as well as for $\mu \to \infty$ ($\ln\mu \to \infty$). The asymptotic values correspond to the problems for a single layer of the thickness $2h_1$ having a crack at its midplane with a free or clamped outer boundaries respectively. For the case of clamped boundaries when $\mu \to \infty$ from (37), (49) and (51) follows that

$$\hat{K}_\infty \equiv \lim_{\mu \to \infty} \hat{K}\left(\mu, \frac{h_2}{h_1}, \frac{l}{h_1}\right) = \exp(A) \tag{53}$$

where

$$A = \frac{1}{\pi}\int_0^\infty \frac{\ln[\tanh(h_1 b\xi)]}{\xi^2+1}\,d\xi. \tag{54}$$

Formulas (53), (54) are in agreement with the results obtained by Ryvkin and Banks-Sills (1994) for a crack at the interface between an elastic strip and a dissimilar elastic half-plane. The second boundary of the strip is clamped and the loading is the same as (34).

When the half-plane becomes absolutely rigid then from the symmetry one gets the considered limiting case of the layer with clamped boundaries. Correspondingly, setting $\mu_2 = \infty$ in eqns (58), (60) of the above reference gives the same expression for the normalized stress intensity factor $\hat{K}_\infty$ as (53), (54). Note that a factor $1/\pi$ is missing in their eqn (58) before the integral. The value of A may be expressed through Γ-functions. Employing the identity

$$p \cot(\pi p) = \frac{\Gamma(1+p)\Gamma(1-p)}{\Gamma(1/2+p)\Gamma(1/2-p)}$$

after some manipulations one obtains

$$A = \left[\frac{\frac{bh_1}{\pi}\Gamma\left(\frac{1}{2}+\frac{bh_1}{\pi}\right)}{\Gamma\left(1+\frac{bh_1}{\pi}\right)} \right]^{1/2} \tag{55}$$

For the second limiting case with free boundaries, setting $\mu \to 0$ gives

$$\hat{K}_0 \equiv \lim_{\mu \to 0} \hat{K}\left(\mu, \frac{h_2}{h_1}, \frac{h_1}{l}\right) = \exp(-A). \tag{56}$$

So, the normalized stress intensity factors for a cracked layer in the cases of the clamped and free boundaries are reciprocals

$$\hat{K}_\infty = \hat{K}_0^{-1}.$$

Central symmetry of the curves which is observed in Fig. 3 means that such a relation is general and true not only for the limiting cases

$$\hat{K}\left(\mu, \frac{h_2}{h_1}, \frac{h_1}{l}\right) = \hat{K}^{-1}\left(\frac{1}{\mu}, \frac{h_2}{h_1}, \frac{h_1}{l}\right), \quad \mu \in (0, \infty). \tag{57}$$

Thus, exchanging the inner and outer materials, which inverts the shear moduli ratio $\mu = \mu_2/\mu_1$, inverts also the normalized stress intensity factor. Analytically this follows from the identity $B(t, \mu)B(t, 1/\mu) \equiv 1$ which can be proven from (37). Formula (57) meets the result obtained by Sih and Chen (1981). They considered a finite length crack under constant shear loading in a layer sandwiched between two half-spaces and derived an analytical expression for the stress intensity factor in the limiting case when the ratio of layer thickness to crack length tends to 0.

4.2. *Energy release rate*

Investigation of the influence of loading length on the fracture characteristics of the problem will be performed in terms of energy release rate G. Applying the formula for the local work at the crack tip

$$G = \frac{1}{a} \lim_{a \to 0} \int_0^a \tau_1^{(0)}(a, -h_1)\Delta w(a)\, dx, \tag{58}$$

where $\Delta w(a) = 2w_1^{(0)}(a, -h_1)$ is the crack opening displacement, and employing the asymptotes (46), (47) one obtains

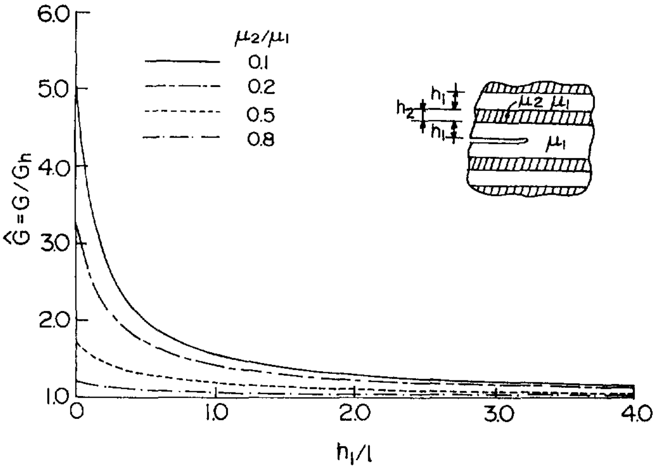

Fig. 4. Graph of the normalized energy release rate $\hat{G}$ vs non-dimensional load distribution parameter h_1/l for the different shear moduli ratio with $h_2/h_1 = 5$.

$$G = \frac{[q_0 N^-(b)]^2}{\mu_1} = \frac{K^2}{2\mu_1} \tag{59}$$

as it should be. Using the value of energy release rate in the reference problem for the crack in the homogeneous body, with the properties of the inner material

$$G_h = \frac{K_h^2}{2\mu_1} = \frac{q_0^2 l}{\mu_1} \tag{60}$$

to normalize the result, gives the expression for the non-dimensional energy release rate for the crack in the layered medium

$$\hat{G} = \frac{G}{G_h} = [N_2^-(b)]^2. \tag{61}$$

Graphs of function $\hat{G}(h_1/l)$ for different shear moduli ratio $\mu = 0.1,\ 0.2,\ 0.5,\ 0.8$ with $h_2/h_1 = 5$ are presented in Fig. 4. From eqns (51), (61) it follows that

$$\hat{K} = \sqrt{\hat{G}}. \tag{62}$$

Consequently the reciprocal property is also valid for the normalized energy release rate

$$\hat{G}(\mu) = \hat{G}^{-1}(1/\mu), \quad \mu \in (0, \infty). \tag{63}$$

For this reason only the curves for the values of $\mu \in (0, 1)$ are presented. Since, as it is seen from (37) and (49), $N_2^+(b) \to 1$ for $h_1/l \to \infty$, irrespective of μ, all the curves have the same asymptote. This is intuitively clear since with decreasing loading decay length l the stress field perturbation produced by the applied loading tends to be localized inside the inner layer which becomes infinitely thick in the l-scale. Consequently, $G \to G_h$ for all μ and in accordance with (61) $\hat{G} \to 1$. The limiting value of the energy release rate for the opposite case of the infinite long loading when $h_1/l \to 0$ is expressed through the magnitude $N_2^+(0)$. Deriving it from (41) by the use of the Sokhotski-Plemelj formula one obtains

$$\lim_{h_1/l \to 0} \hat{G} = \sqrt{\frac{\mu h_1 + h_2}{\mu(h_1 + \mu h_2)}}. \tag{64}$$

It is worthwhile to present also the limiting value of the stress intensity factor for the infinite long loading. From (62) immediately follows that

$$\lim_{h_1/l \to 0} \hat{K} = \left[\frac{\mu h_1 + h_2}{\mu(h_1 + \mu h_2)}\right]^{1/4}. \tag{65}$$

The asymptotic results (64), (65) may be confirmed in the following manner. When the loading decay length tends to infinity, significantly exceeding the layers thicknesses, the far field displacements of the crack faces tend to those of a crack in an anisotropic homogeneous body possessing the effective elastic properties and subjected to the same loading. Then, viewing the energy release rate as calculated through the work of the external forces, one concludes

$$\lim_{h_1/l \to 0} \hat{G} = \hat{G}_a, \tag{66}$$

where $\hat{G}_a$ is the value of energy release rate for the crack in the anisotropic homogeneous body, normalized in accordance with (60), (61). Values of the effective axial and tangential elastic shear moduli of the periodically laminated transversely isotropic medium were determined by Postma (1955)

$$\mu_A^* = \frac{\mu(h_1 + h_2)}{\mu h_1 + h_2}\mu_1, \quad \mu_T^* = \frac{h_1 + \mu h_2}{h_1 + h_2}\mu_1. \tag{67}$$

A fundamental study of cracks in anisotropic elastic bodies was accomplished by Sih *et al.* (1955). Using their results it is possible to show that the stress intensity factor for the crack in the considered anisotropic homogeneous body will be the same as in the reference problem for the isotropic one. Employing next the formula relating stress intensity factor and energy release rate in the transversely anisotropic body for calculating $\hat{G}_a$ leads to the expression coinciding with the right hand side of (64).

Hence the averaging anisotropic model gives a true asymptotic result for the energy release rate but is unfit for calculating the stress intensity factor defined by the local stress field. Formula (65) for the stress intensity factor may be confirmed by the help of the eigen-solution for the considered laminated medium with the semi-infinite crack. Following Ryvkin *et al.* it is possible to derive this solution from the results for the specific loading obtained in Section 3. Assuming that

$$q_0 = \frac{C}{\sqrt{l}} \tag{68}$$

and taking a limit for $l \to \infty$ one gets the solution for an infinite long loading with an infinite small amplitude, i.e., the eigen-solution. In front of the crack the near and the remote stress distributions in this solution have near and far asymptotes respectively. These asymptotes are characterized by the near K_n and the far K_f stress intensity factors, connected by the relation

$$\frac{K_n}{K_f} = \left[\frac{\mu h_1 + h_2}{\mu(h_1 + \mu h_2)}\right]^{1/4}, \tag{69}$$

which agrees with (65).

4.3. *Comparison with the sandwich model*

The obtained exact solution of the initial problem of a crack in a layered medium gives an opportunity to examine the accuracy of some approximate models widely used in fracture analysis of laminates. Of course, this comparison may be carried out only for periodically laminated bimaterial systems but the obtained results will supply valuable information also for the modeling of more complicated multilayered composites. Some estimations for the simplest model, when elastic properties of the half-spaces sandwiching the cracked layer of one material are associated with the properties of the second material of the bimaterial laminate, may be obtained directly from Fig. 3. The curve for $h_2/h_1 = \infty$ there may be viewed as representing the solution for the approximate model. Comparing it, for example, with the curve for $h_2/h_1 = 1$ one can conclude that for the considered loading decay length $h_1/l = 0.5$ a relative error in calculation of the stress intensity factor will not exceed 10% over a whole range of the shear moduli ratio for the cases when $h_2 > h_1$.

In a more precise sandwich model it is supposed that properties of the isotropic half-spaces are the averages over a large number of layers of the actual laminate. The limitations of this model may also be studied by means of the obtained solution. Suppose that for a specific combination of problem parameters h_2/h_1, μ_2/μ_1, h_1/l an average shear modulus μ_2^{av} of the half-spaces has been chosen in such a way that the stress intensity factor derived from the approximate model is the same as in the exact multilayered one. The natural requirement for the approximate model is that it is must be valid for different loading characteristics. To investigate this issue a numerical procedure was developed based on the comparison between the solution for the exact model with the actual value of h_2 and the solution for $h_2 = \infty$. The non-dimensional value $\mu^{av} = \mu_2^{av}/\mu_1$ was determined from the condition of equality of the stress intensity factors. Repeating this procedure for different values of the loading parameter h_1/l allowed to examine the dependence of μ^{av} on h_1/l. Graphical results for three different values of shear moduli ratio in an actual laminate $\mu = 0.1$, 0.3 and 0.5 with $h_2/h_1 = 1$ are depicted in Fig. 5. The limiting values of μ^{av} for the case of infinite long loading $h_1/l \to 0$ may be found analytically. Setting $h_2 \to \infty$ in (65) and equating the obtained result to the exact value of the stress intensity factor with the finite h_2 one obtains

$$\lim_{h_1/l \to 0} \mu^{av} = \left[\frac{\mu(h_1 + \mu h_2)}{\mu h_1 + h_2}\right]^{1/2}. \tag{70}$$

Note that this last formula is of particular interest. It represents the expression for the average shear modulus in the sandwich approximation of the periodically layered

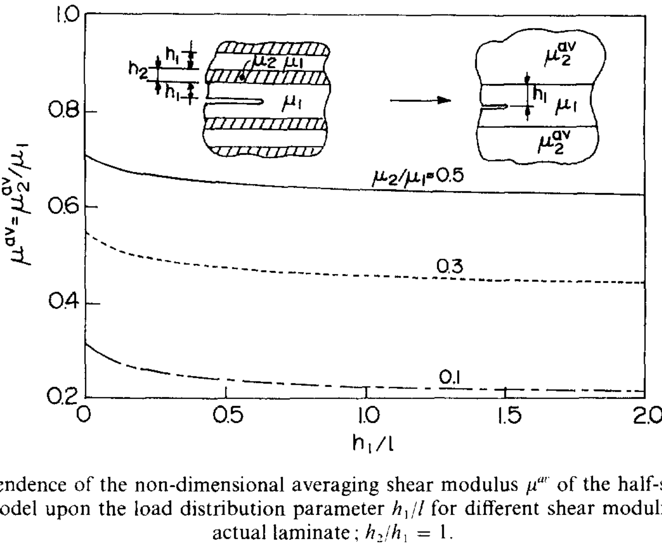

Fig. 5. Dependence of the non-dimensional averaging shear modulus μ^{av} of the half-spaces in the sandwich model upon the load distribution parameter h_1/l for different shear moduli ratio in the actual laminate; $h_2/h_1 = 1$.

composite, giving the exact value of stress intensity factor in the case of a sufficiently long loading. For the considered example with equal thicknesses

$$\lim_{h_1/l \to 0} \mu^{av} = \sqrt{\mu}, \tag{71}$$

or, in the dimensional form, $\mu_2^{av} \to \sqrt{\mu_1 \mu_2}$. For all the curves a significant variation of averaging shear modulus in the region $0 < h_1/l < 0.5$ is observed. Therefore, employing the sandwich model will be of limited application for the mentioned values of the parameter associated with the loading decay length. Finally, it may be noted that studying the dependence $\mu^{av}(h_1/l)$ for the case of short loadings, when $2 < h_1/l$, is of little interest. In this case, as it follows from Fig. 4 and (62), the value of stress intensity factor does not essentially differ from that for the crack in a homogeneous space.

5. CONCLUDING REMARKS

The presented closed form solution for the problem of a Mode III crack in a periodically layered bimaterial media allowed to investigate the influence of the problem parameters on the fracture characteristics.

It was found, as expected, that the stress intensity factor decreases (increases) monotonically with respect to its value for a homogeneous space with reinforcing (weakening) of the media, caused either by increasing (decreasing) the outer layers shear modulus or by increasing their thicknesses. For the considered exponentially decaying load it was shown that exchanging the materials of the layers leads to the inversion of the values of normalized stress intensity factor and energy release rate. Consequently, normalized stress intensity factors for the limiting cases of bilayered media when outer layers have disappeared or have become absolutely rigid, are found to be reciprocals.

The validity of the approximate models with reduced number of layers was examined. For this goal we have compared the obtained solution for the composite with an infinite number of layers with the solution for a crack in a layer sandwiched between two half-spaces. For the case when the characteristic loading decay length is of the order of cracked layer thickness or less, the following result is recovered. If the outer layers are thicker then the inner ones, then the calculation of the stress intensity factor by use of the sandwich approximation with the material of the half-spaces associated with the material of the outer layers leads to a less than about 10% error. Employing the model with the effective shear modulus of the half-spaces improves the accuracy of the sandwich model, but for a sufficiently long loading its application becomes questionable. On the other hand, in the limiting case of an infinitely long loading simple formula for the effective shear modulus may be employed.

From symmetry considerations all the obtained results may be used in the study of antiplane debonding of multilayered laminates from a rigid substrate.

It may be noted that the based on the discrete Fourier transform mathematical technique, developed by Nuller (1981) and implemented in the present paper, is found to be a convenient tool for the study of fracture of periodically layered laminates. It is also applicable in more complicated cases such as Mode I and Mode II cracks, steady state crack propagation and cracks in laminates of finite thickness.

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